Cambridge Assessment
Admissions Testing

Sixth Term Examination Papers **9475**
MATHEMATICS 3 Morning
Thursday 21 June 2018 Time: 3 hours

2682044645

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and circle the paper number in the spaces provided on the answer booklet.

Make sure you fill in page 1 **AND** page 3 of the answer booklet with your details.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

This question paper consists of 10 printed pages and 2 blank pages.

PV1

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Section A: Pure Mathematics

1 **(i)** The function f is given by

$$\mathrm{f}(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

Find the stationary point of the curve $y = \mathrm{f}(\beta)$ and sketch the curve.

Sketch also the curve $y = \mathrm{g}(\beta)$, where

$$\mathrm{g}(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

(ii) Let u and v be the roots of the equation

$$x^2 + \alpha x + \beta = 0\,,$$

where $\beta \neq 0$. Obtain expressions in terms of α and β for $u + v + \dfrac{1}{uv}$ and $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

(iii) Given that $u + v + \dfrac{1}{uv} = -1$, and that u and v are real, show that $\dfrac{1}{u} + \dfrac{1}{v} + uv \leqslant -1$.

(iv) Given instead that $u + v + \dfrac{1}{uv} = 3$, and that u and v are real, find the greatest value of $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

2 The sequence of functions $y_0, y_1, y_2, \ldots$ is defined by $y_0 = 1$ and, for $n \geqslant 1$,

$$y_n = (-1)^n \frac{1}{z} \frac{\mathrm{d}^n z}{\mathrm{d}x^n}\,,$$

where $z = \mathrm{e}^{-x^2}$.

(i) Show that $\dfrac{\mathrm{d}y_n}{\mathrm{d}x} = 2xy_n - y_{n+1}$ for $n \geqslant 1$.

(ii) Prove by induction that, for $n \geqslant 1$,

$$y_{n+1} = 2xy_n - 2ny_{n-1}\,.$$

Deduce that, for $n \geqslant 1$,

$$y_{n+1}^2 - y_n y_{n+2} = 2n\big(y_n^2 - y_{n-1}y_{n+1}\big) + 2y_n^2\,.$$

(iii) Hence show that $y_n^2 - y_{n-1}y_{n+1} > 0$ for $n \geqslant 1$.

3 Show that the second-order differential equation

$$x^2y'' + (1-2p)x\,y' + (p^2-q^2)\,y = \mathrm{f}(x)\,,$$

where p and q are constants, can be written in the form

$$x^a\big(x^b(x^cy)'\big)' = \mathrm{f}(x)\,, \qquad (*)$$

where a, b and c are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = 0$$

in the different cases that arise according to the values of p and q.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of p and n.

4 The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at P can be written as

$$bx - ay\sin\theta = ab\cos\theta\,.$$

(i) This tangent meets the lines $\dfrac{x}{a} = \dfrac{y}{b}$ and $\dfrac{x}{a} = -\dfrac{y}{b}$ at S and T, respectively.

How is the mid-point of ST related to P?

(ii) The point $Q(a\sec\phi, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at P and Q are perpendicular. These two tangents intersect at (x, y).

Obtain expressions for x^2 and y^2 in terms of a, θ and ϕ.

Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.

5 The real numbers a_1, a_2, a_3, ... are all positive. For each positive integer n, A_n and G_n are defined by

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad G_n = \big(a_1 a_2 \cdots a_n\big)^{1/n}.$$

(i) Show that, for any given positive integer k,

$$(k+1)(A_{k+1} - G_{k+1}) \geqslant k(A_k - G_k)$$

if and only if

$$\lambda_k^{k+1} - (k+1)\lambda_k + k \geqslant 0\,,$$

where $\lambda_k = \left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$.

(ii) Let

$$\mathrm{f}(x) = x^{k+1} - (k+1)x + k\,,$$

where $x > 0$ and k is a positive integer. Show that $\mathrm{f}(x) \geqslant 0$ and that $\mathrm{f}(x) = 0$ if and only if $x = 1$.

(iii) Deduce that:

(a) $A_n \geqslant G_n$ for all n;

(b) if $A_n = G_n$ for some n, then $a_1 = a_2 = \cdots = a_n$.

6 **(i)** The distinct points A, Q and C lie on a straight line in the Argand diagram, and represent the distinct complex numbers a, q and c, respectively. Show that $\dfrac{q-a}{c-a}$ is real and hence that $(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$.

Given that $aa^* = cc^* = 1$, show further that

$$q + acq^* = a + c\,.$$

(ii) The distinct points A, B, C and D lie, in anticlockwise order, on the circle of unit radius with centre at the origin (so that, for example, $aa^* = 1$). The lines AC and BD meet at Q. Show that

$$(ac-bd)q^* = (a+c) - (b+d)\,,$$

where b and d are complex numbers represented by the points B and D respectively, and show further that

$$(ac-bd)(q+q^*) = (a-b)(1+cd) + (c-d)(1+ab)\,.$$

(iii) The lines AB and CD meet at P, which represents the complex number p. Given that p is real, show that $p(1+ab) = a+b$. Given further that $ac-bd \neq 0$, show that

$$p(q+q^*) = 2\,.$$

7 **(i)** Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then

$$\frac{(\cot\theta + \mathrm{i})^{2n+1} - (\cot\theta - \mathrm{i})^{2n+1}}{2\mathrm{i}} = \frac{\sin(2n+1)\,\theta}{\sin^{2n+1}\theta},$$

for any positive integer n.

Deduce that the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right)$$

where $m = 1, 2, \ldots, n$.

(ii) Hence show that

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n\,(2n-1)}{3}.$$

(iii) Given that $0 < \sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta.$$

Hence show that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

8 In this question, you should ignore issues of convergence.

(i) Let

$$I = \int_0^1 \frac{\mathrm{f}(x^{-1})}{1+x}\,\mathrm{d}x,$$

where $\mathrm{f}(x)$ is a function for which the integral exists.

Show that

$$I = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\mathrm{f}(y)}{y(1+y)}\,\mathrm{d}y$$

and deduce that, if $\mathrm{f}(x) = \mathrm{f}(x+1)$ for all x, then

$$I = \int_0^1 \frac{\mathrm{f}(x)}{1+x}\,\mathrm{d}x.$$

(ii) The *fractional part*, $\{x\}$, of a real number x is defined to be $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example $\{3.2\} = 0.2$ and $\{3\} = 0$.

Use the result of part (i) to evaluate

$$\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x.$$

Section B: Mechanics

9 A particle P of mass m is projected with speed u_0 along a smooth horizontal floor directly towards a wall. It collides with a particle Q of mass km which is moving directly away from the wall with speed v_0. In the subsequent motion, Q collides alternately with the wall and with P. The coefficient of restitution between Q and P is e, and the coefficient of restitution between Q and the wall is 1.

Let u_n and v_n be the velocities of P and Q, respectively, towards the wall after the nth collision between P and Q.

(i) Show that, for $n \geqslant 2$,

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0\,. \qquad (*)$$

(ii) You are now given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, and that the solution of $(*)$ is of the form

$$u_n = A\left(\tfrac{7}{10}\right)^n + B\left(\tfrac{5}{7}\right)^n \qquad (n \geqslant 0)\,,$$

where A and B are independent of n. Find expressions for A and B in terms of u_0 and v_0.

Show that, if $0 < 6u_0 < v_0$, then u_n will be negative for large n.

10 A uniform disc with centre O and radius a is suspended from a point A on its circumference, so that it can swing freely about a horizontal axis L through A. The plane of the disc is perpendicular to L. A particle P is attached to a point on the circumference of the disc. The mass of the disc is M and the mass of the particle is m.

In equilibrium, the disc hangs with OP horizontal, and the angle between AO and the downward vertical through A is β. Find $\sin\beta$ in terms of M and m and show that

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}\,.$$

The disc is rotated about L and then released. At later time t, the angle between OP and the horizontal is θ; when P is higher than O, θ is positive and when P is lower than O, θ is negative. Show that

$$\tfrac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta\,(1-\cos\theta)$$

is constant during the motion, where I is the moment of inertia of the disc about L.

Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$, show that the period of small oscillations is

$$3\pi\sqrt{\frac{3a}{5g}}\,.$$

11 A particle is attached to one end of a light inextensible string of length b. The other end of the string is attached to a fixed point O. Initially the particle hangs vertically below O. The particle then receives a horizontal impulse.

The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is α, at which time it becomes slack. Express V, the speed of the particle when the string becomes slack, in terms of b, g and α.

Show that the string becomes taut again a time T later, where

$$gT = 4V \sin\alpha\,,$$

and that just before this time the trajectory of the particle makes an angle β with the horizontal where $\tan\beta = 3\tan\alpha$.

When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}\,.$$

Section C: Probability and Statistics

12 A random process generates, independently, n numbers each of which is drawn from a uniform (rectangular) distribution on the interval 0 to 1. The random variable Y_k is defined to be the kth smallest number (so there are $k-1$ smaller numbers).

(i) Show that, for $0 \leqslant y \leqslant 1$,

$$\mathrm{P}\big(Y_k \leqslant y\big) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m} . \qquad (*)$$

(ii) Show that

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

and obtain a similar expression for $(n-m)\dbinom{n}{m}$.

Starting from $(*)$, show that the probability density function of Y_k is

$$n\binom{n-1}{k-1} y^{k-1} (1-y)^{n-k} .$$

Deduce an expression for $\displaystyle\int_0^1 y^{k-1}(1-y)^{n-k}\,\mathrm{d}y$.

(iii) Find $\mathrm{E}(Y_k)$ in terms of n and k.

13 The random variable X takes only non-negative integer values and has probability generating function $\mathrm{G}(t)$. Show that

$$\mathrm{P}(X = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6\ \ldots) = \frac{1}{2}\big(\mathrm{G}(1) + \mathrm{G}(-1)\big).$$

You are now given that X has a Poisson distribution with mean λ. Show that

$$\mathrm{G}(t) = \mathrm{e}^{-\lambda(1-t)}\,.$$

(i) The random variable Y is defined by

$$\mathrm{P}(Y = r) = \begin{cases} k\mathrm{P}(X = r) & \text{if } r = 0,\ 2,\ 4,\ 6,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where k is an appropriate constant.

Show that the probability generating function of Y is $\dfrac{\cosh \lambda t}{\cosh \lambda}$.

Deduce that $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

(ii) The random variable Z is defined by

$$\mathrm{P}(Z = r) = \begin{cases} c\mathrm{P}(X = r) & \text{if } r = 0,\ 4,\ 8,\ 12,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where c is an appropriate constant.

Is $\mathrm{E}(Z) < \lambda$ for all positive values of λ?

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